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Sets and Categories – Solutions

End-of-chapter worksheet for Aluffi's *Chapter 0*, Chapter 1

Throughout this worksheet, the phrase "C is a category with products and coproducts" means that C is a category and that for every pair of objects A and B of C there is an object $A \times B$ satisfying the universal property for products and an object $A \amalg B$ satisfying the universal property for coproducts.

Exercise 1. Let C be a category with products and coproducts and let 0 be an initial object in C.

(a) Show that $A \amalg 0 \cong A$ for every object A of C.

(b) Show that $A \times 0 \cong 0$ for every object A of C if and only if every morphism in C with codomain 0 is an isomorphism.

(c) Explain how this applies to the category Set.

Solution. **(a)** We show that A itself, together with the canonical maps

$$\mathrm{id}_A \colon A \longrightarrow A \qquad \text{and} \qquad !_A \colon 0 \longrightarrow A,$$

satisfies the universal property of the coproduct of A and 0. Here $!_A$ is the unique morphism from the initial object 0 to A.

Suppose X is any object of C and we are given morphisms $f \colon A \to X$ and $g \colon 0 \to X$. Take $h := f \colon A \to X$. The situation is

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A"] \arrow[dr, "f"'] & A \arrow[d, dashed, "h"] & 0 \arrow[l, "!_A"'] \arrow[dl, "g"] \\
& X &
\end{tikzcd}$$

Then $h \circ \mathrm{id}_A = f$, and $h \circ !_A = f \circ !_A$ is a morphism $0 \to X$. But 0 is initial, so $\mathrm{Hom}_{\mathsf{C}}(0, X)$ has exactly one element. Therefore $f \circ !_A = g$ automatically.

Uniqueness: if $h' \colon A \to X$ also satisfies $h' \circ \mathrm{id}_A = f$ and $h' \circ !_A = g$, then $h' = h' \circ \mathrm{id}_A = f = h$.

Hence A together with the maps id_A and $!_A$ is a coproduct of A and 0, and so $A \amalg 0 \cong A$.

(b) ($\Leftarrow$) Suppose every morphism in C with codomain 0 is an isomorphism. The product projection $\pi_0 \colon A \times 0 \to 0$ is then an isomorphism, so $A \times 0 \cong 0$.

($\Rightarrow$) Suppose $A \times 0 \cong 0$ for every object A. Let $f \colon B \to 0$ be any morphism with codomain 0; we show f is an isomorphism.

By the universal property of $B \times 0$ applied to the pair $(\mathrm{id}_B \colon B \to B,\ f \colon B \to 0)$, there is a unique morphism

$$\phi \colon B \longrightarrow B \times 0 \qquad \text{with} \qquad \pi_B \circ \phi = \mathrm{id}_B \quad \text{and} \quad \pi_0 \circ \phi = f.$$

In a diagram:

$$\begin{tikzcd}
& B \arrow[dl, "\mathrm{id}_B"'] \arrow[d, dashed, "\phi"] \arrow[dr, "f"] & \\
B & B \times 0 \arrow[l, "\pi_B"] \arrow[r, "\pi_0"'] & 0
\end{tikzcd}$$

Since $B \times 0 \cong 0$, the object $B \times 0$ is itself initial. In particular $\mathrm{Hom}_{\mathsf{C}}(B \times 0, B \times 0) = \{\mathrm{id}_{B \times 0}\}$, so the composite

$$\phi \circ \pi_B \colon B \times 0 \longrightarrow B \times 0$$

must equal $\mathrm{id}_{B \times 0}$. Combined with $\pi_B \circ \phi = \mathrm{id}_B$, this shows ϕ and π_B are mutually inverse, so $B \cong B \times 0$.

Similarly, $\mathrm{Hom}_{\mathsf{C}}(B \times 0, 0)$ is a singleton (because $B \times 0$ is initial), and any morphism between two initial objects is an isomorphism, so the projection $\pi_0 \colon B \times 0 \to 0$ is itself an isomorphism.

Finally, $f = \pi_0 \circ \phi$ is a composition of two isomorphisms, hence an isomorphism.

(c) In Set, the initial object is $\emptyset$. A morphism $A \to \emptyset$ is a function from A to the empty set, which exists if and only if $A = \emptyset$, in which case the morphism is $\mathrm{id}_\emptyset$. Hence every morphism in Set with codomain $\emptyset$ is an isomorphism, and part (b) gives

$$A \times \emptyset \cong \emptyset \qquad \text{for every set } A,$$

recovering the familiar fact $A \times \emptyset = \emptyset$. Part (a) gives $A \amalg \emptyset \cong A$, recovering the equally familiar fact that adjoining nothing to A leaves A unchanged.

Exercise 2.

(a) Prove that in Set, products distribute over coproducts up to isomorphism:

$$(A \amalg B) \times C \;\cong\; (A \times C) \amalg (B \times C)$$

for all objects A, B, C of Set.

(b) Find a category with products and coproducts for which the above distributive law does *not* hold.

Solution. **(a)** Model the coproduct in Set as the disjoint union

$$A \amalg B = (A \times \{0\}) \cup (B \times \{1\}),$$

with inclusions $\iota_A(a) = (a, 0)$ and $\iota_B(b) = (b, 1)$. Define

$$\Phi\colon (A \amalg B) \times C \longrightarrow (A \times C) \amalg (B \times C)$$

by

$$\Phi\big((a,0),\, c\big) = \big((a,c),\, 0\big), \qquad \Phi\big((b,1),\, c\big) = \big((b,c),\, 1\big).$$

This is well-defined because every element of $(A \amalg B) \times C$ has its first coordinate in exactly one of the two summands of $A \amalg B$.

Define Ψ in the opposite direction by

$$\Psi\big((a,c),\, 0\big) = \big((a,0),\, c\big), \qquad \Psi\big((b,c),\, 1\big) = \big((b,1),\, c\big).$$

A direct check on each summand shows $\Psi \circ \Phi = \mathrm{id}$ and $\Phi \circ \Psi = \mathrm{id}$. Hence Φ is a bijection, which in Set is the same as an isomorphism.

(b) In the category of abelian groups Ab, finite products and finite coproducts coincide: both are given by the direct sum. Thus the distributive law would read

$$(A \oplus B) \oplus C \;\cong\; (A \oplus C) \oplus (B \oplus C).$$

Taking $A = B = C = \mathbb{Z}$ gives $\mathbb{Z}^3$ on the left and $\mathbb{Z}^4$ on the right, which are not isomorphic as abelian groups (compare ranks). So Ab is a category with products and coproducts in which products do not distribute over coproducts.

Exercise 3. Let $S = \{0, a, b, c, x, y, z\}$ and consider the partial ordering on S given by the relations

$$a \le x, \quad b \le x, \quad a \le y, \quad b \le y, \quad c \le y, \quad c \le z,$$

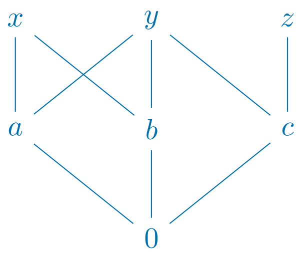
together with $0 \le$ everything.

(a) Draw a poset diagram for $\le$ as follows. Place 0 alone at the bottom; place a, b, c in a row above 0; place x, y, z in a row above a, b, c. Draw an edge between elements in adjacent rows whenever the lower element is $\le$ the higher one.

(b) As in Example 3.3 (Aluffi, p. 20), we obtain a category whose objects are the elements of S and whose morphisms are pairs (u, v) for those $u, v \in S$ with $u \le v$. Show that products in this category fail to be associative: $x \times (y \times z)$ exists, but $(x \times y) \times z$ does not.

(c) Explain why this does *not* contradict the conclusion of Exercise 5.9 (Aluffi, p. 38).

Solution. **(a)** The poset diagram is



(b) In the category associated to a poset $(S, \leq)$, the product of two objects u and v is their meet (greatest lower bound) when it exists. We compute each side.

First we compute $x \times (y \times z)$. The set of common lower bounds of y and z in S is $\{u \in S : u \leq y \text{ and } u \leq z\} = \{0, c\}$, which has greatest element c. Hence $y \times z = c$. The common lower bounds of x and c are $\{u : u \leq x \text{ and } u \leq c\} = \{0\}$, so $x \times (y \times z) = x \times c = 0$.

Now consider $(x \times y) \times z$. The common lower bounds of x and y are $\{u : u \leq x \text{ and } u \leq y\} = \{0, a, b\}$. But a and b are incomparable, so this set has no greatest element. Hence $x \times y$ does not exist in this poset, and therefore $(x \times y) \times z$ does not exist either.

So $x \times (y \times z)$ exists and equals 0, while $(x \times y) \times z$ does not exist. Associativity fails because one side does not exist.

(c) Aluffi's Exercise 5.9 says that *whenever* the relevant products all exist, there is a canonical isomorphism $A \times (B \times C) \cong (A \times B) \times C$. In our example the product $x \times y$ does not exist, so the hypothesis of Exercise 5.9 fails, and its conclusion does not apply. There is no contradiction: associativity of products is a conditional statement that presumes the relevant products exist.

Exercise 4. Suppose G and H are groups. Let $G \amalg H$ denote their disjoint union and let $G \times H$ denote their Cartesian product. (That is, $G \amalg H$ and $G \times H$ are the coproduct and product of the underlying sets, respectively.)

(a) Prove that **there is no** group operation on $G \amalg H$ for which the canonical inclusions $G \to G \amalg H$ and $H \to G \amalg H$ are group homomorphisms.

(b) Prove that **there is** a group operation on $G \times H$ for which the canonical projections $G \times H \to G$ and $G \times H \to H$ are group homomorphisms.

(c) The group operation in part (b) makes $G \times H$ the product in the category of groups. Take a guess: does the category of groups have coproducts?

Solution. **(a)** Suppose, for contradiction, that $G \amalg H$ carries a group operation for which the inclusions

$$\iota_G \colon G \longrightarrow G \amalg H \qquad \text{and} \qquad \iota_H \colon H \longrightarrow G \amalg H$$

are group homomorphisms. Any group homomorphism sends the identity to the identity, so

$$\iota_G(e_G) = e_{G \amalg H} \qquad \text{and} \qquad \iota_H(e_H) = e_{G \amalg H},$$

which forces $\iota_G(e_G) = \iota_H(e_H)$. But $G \amalg H$ is the *disjoint* union of G and H: the images of ι_G and ι_H are by definition disjoint subsets of $G \amalg H$. In particular $\iota_G(e_G) \neq \iota_H(e_H)$, a contradiction.

(b) Define the operation on $G \times H$ componentwise:

$$(g_1, h_1) \cdot (g_2, h_2) := (g_1 g_2,\ h_1 h_2).$$

This is associative because multiplication in G and in H is associative, has identity (e_G, e_H), and admits inverses $(g, h)^{-1} = (g^{-1}, h^{-1})$. Hence $(G \times H, \cdot)$ is a group.

The projection $\pi_G \colon G \times H \to G$, $(g, h) \mapsto g$, satisfies

$$\pi_G\big((g_1, h_1)(g_2, h_2)\big) = \pi_G(g_1 g_2, h_1 h_2) = g_1 g_2 = \pi_G(g_1, h_1)\,\pi_G(g_2, h_2),$$

so π_G is a group homomorphism. The same calculation, with the second coordinate, shows $\pi_H \colon G \times H \to H$ is a group homomorphism as well.

(c) Yes. The category of groups does have coproducts; the coproduct of G and H is the *free product* $G * H$. Part (a) shows that the underlying set of the coproduct in Grp *cannot* be the disjoint union of the underlying sets of G and H, so the forgetful functor $\mathsf{Grp} \to \mathsf{Set}$ does not preserve coproducts (in contrast to part (b), where it does preserve products).