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Sets and Categories

End-of-chapter worksheet for Aluffi's *Chapter 0*, Chapter 1

Throughout this worksheet, the phrase "C is a category with products and coproducts" means that C is a category and that for every pair of objects A and B of C there is an object $A \times B$ satisfying the universal property for products and an object $A \amalg B$ satisfying the universal property for coproducts.

Exercise 1. Let C be a category with products and coproducts and let 0 be an initial object in C.

(a) Show that $A \amalg 0 \cong A$ for every object A of C.

(b) Show that $A \times 0 \cong 0$ for every object A of C if and only if every morphism in C with codomain 0 is an isomorphism.

(c) Explain how this applies to the category Set.

Exercise 2.

(a) Prove that in Set, products distribute over coproducts up to isomorphism:

$$(A \amalg B) \times C \;\cong\; (A \times C) \amalg (B \times C)$$

for all objects A, B, C of Set.

(b) Find a category with products and coproducts for which the above distributive law does *not* hold.

Exercise 3. Let $S = \{0, a, b, c, x, y, z\}$ and consider the partial ordering on S given by the relations

$$a \le x, \quad b \le x, \quad a \le y, \quad b \le y, \quad c \le y, \quad c \le z,$$

together with $0 \le$ everything.

(a) Draw a poset diagram for $\le$ as follows. Place 0 alone at the bottom; place a, b, c in a row above 0; place x, y, z in a row above a, b, c. Draw an edge between elements in adjacent rows whenever the lower element is $\le$ the higher one.

(b) As in Example 3.3 (Aluffi, p. 20), we obtain a category whose objects are the elements of S and whose morphisms are pairs (u, v) for those $u, v \in S$ with $u \leq v$. Show that products in this category fail to be associative: $x \times (y \times z)$ exists, but $(x \times y) \times z$ does not.

(c) Explain why this does *not* contradict the conclusion of Exercise 5.9 (Aluffi, p. 38).

Exercise 4. Suppose G and H are groups. Let $G \amalg H$ denote their disjoint union and let $G \times H$ denote their Cartesian product. (That is, $G \amalg H$ and $G \times H$ are the coproduct and product of the underlying sets, respectively.)

(a) Prove that **there is no** group operation on $G \amalg H$ for which the canonical inclusions $G \to G \amalg H$ and $H \to G \amalg H$ are group homomorphisms.

(b) Prove that **there is** a group operation on $G \times H$ for which the canonical projections $G \times H \to G$ and $G \times H \to H$ are group homomorphisms.

(c) The group operation in part (b) makes $G \times H$ the product in the category of groups. Take a guess: does the category of groups have coproducts?