



# Farey Sums – Solutions

## Number Theory Exercises

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The *Farey sum* of fractions  $\frac{a}{b}$  and  $\frac{c}{d}$  is defined by the formula

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}.$$

The symbol  $\oplus$  is a reminder that the Farey sum is not ordinary addition:  $\frac{0}{1} \oplus \frac{1}{1} = \frac{1}{2}$ , not 1. It also depends on the representatives chosen:  $\frac{0}{2} \oplus \frac{1}{1} = \frac{1}{3}$ .

Start with  $\frac{0}{1}$  and  $\frac{1}{1}$ . At each stage, insert between every pair of neighboring fractions their Farey sum. The first five stages, with each stage's new fractions in bold:

$$\text{Stage 1: } \frac{0}{1}, \frac{1}{1}$$

$$\text{Stage 2: } \frac{0}{1}, \frac{1}{2}, \frac{1}{1}$$

$$\text{Stage 3: } \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}$$

$$\text{Stage 4: } \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}$$

$$\text{Stage 5: } \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}$$

The exercises build toward Exercise 4: every rational number between 0 and 1 eventually appears.

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**Exercise 1.** Show that at each stage in the procedure, the fractions appear in increasing order. (Hint: Induction. Compare each inserted fraction with the neighbor on each side.)

**Solution.** We prove the claim by induction on the stage number  $n$ .

*Base case.* At Stage 1 the fractions are  $\frac{0}{1}, \frac{1}{1}$ , and  $\frac{0}{1} < \frac{1}{1}$ , so the claim holds.

*Inductive step.* Suppose the fractions at Stage  $n$  appear in increasing order, say

$$\frac{p_1}{q_1} < \frac{p_2}{q_2} < \dots < \frac{p_k}{q_k}.$$

At Stage  $n+1$  the Farey sum  $\frac{p_i+p_{i+1}}{q_i+q_{i+1}}$  is inserted between each consecutive pair. We claim that whenever  $\frac{a}{b} < \frac{c}{d}$  with  $b, d > 0$ ,

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.$$

Indeed,  $\frac{a}{b} < \frac{c}{d}$  is equivalent to  $ad < bc$  (since  $b, d > 0$ ). The left inequality  $\frac{a}{b} < \frac{a+c}{b+d}$  is equivalent to  $a(b+d) < b(a+c)$ , which simplifies to  $ad < bc$ ; the right inequality is symmetric.

Applying this lemma to each consecutive pair of Stage  $n$  neighbors gives the Stage  $n+1$  list in strictly increasing order, completing the induction.

**Exercise 2.** Assume  $\frac{a}{b}$  and  $\frac{c}{d}$  are neighbors at some stage of the procedure with  $\frac{a}{b} < \frac{c}{d}$ .

(a) Show  $bc - ad = 1$ . (Hint: Induction. A pair of neighbors at the next stage has one of two forms.)

(b) Show

$$\frac{c}{d} - \frac{a}{b} = \frac{1}{bd}.$$

**Solution.** (a) We prove the identity by induction on the stage number  $n$ .

*Base case.* At Stage 1 the only pair of neighbors is  $\frac{a}{b} = \frac{0}{1}$  and  $\frac{c}{d} = \frac{1}{1}$ , and  $bc - ad = 1 \cdot 1 - 0 \cdot 1 = 1$ .

*Inductive step.* Suppose every pair of neighbors at Stage  $n$  satisfies  $bc - ad = 1$ . Each consecutive pair at Stage  $n+1$  has one of two forms, where  $\frac{a}{b}$  and  $\frac{c}{d}$  are neighbors at Stage  $n$ :

$$\left(\frac{a}{b}, \frac{a+c}{b+d}\right) \quad \text{or} \quad \left(\frac{a+c}{b+d}, \frac{c}{d}\right).$$

For the first pair,

$$b(a+c) - a(b+d) = ab + bc - ab - ad = bc - ad = 1,$$

and for the second,

$$(b+d)c - (a+c)d = bc + cd - ad - cd = bc - ad = 1.$$

The identity therefore holds for every pair of neighbors at Stage  $n+1$ , and by induction at every stage.

(b) Using part (a),

$$\frac{c}{d} - \frac{a}{b} = \frac{bc - ad}{bd} = \frac{1}{bd}.$$

**Exercise 3.** Assume  $\frac{a}{b}$  and  $\frac{c}{d}$  are as in Exercise 2. Let  $\frac{p}{q}$  be a fraction with  $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$ . Show that  $q \geq b+d$ .

Hint:

$$\frac{c}{d} - \frac{a}{b} = \left(\frac{c}{d} - \frac{p}{q}\right) + \left(\frac{p}{q} - \frac{a}{b}\right).$$

**Solution.** Since  $\frac{a}{b} < \frac{p}{q}$  and  $b, q > 0$ , we have  $bp > aq$ . As  $bp - aq$  is a positive integer,  $bp - aq \geq 1$ , so

$$\frac{p}{q} - \frac{a}{b} = \frac{bp - aq}{bq} \geq \frac{1}{bq}.$$

Similarly,  $\frac{p}{q} < \frac{c}{d}$  gives  $cq - dp \geq 1$ , so

$$\frac{c}{d} - \frac{p}{q} \geq \frac{1}{dq}.$$

Using the hint and Exercise 2(b),

$$\frac{1}{bd} = \frac{c}{d} - \frac{a}{b} = \left(\frac{c}{d} - \frac{p}{q}\right) + \left(\frac{p}{q} - \frac{a}{b}\right) \geq \frac{1}{dq} + \frac{1}{bq} = \frac{b+d}{bdq}.$$

Multiplying both sides by  $bdq > 0$  gives  $q \geq b + d$ .

**Exercise 4.** Prove that every rational number between 0 and 1 appears somewhere in the procedure. (Hint: Assume for contradiction that  $\frac{p}{q}$  does not appear at any stage. So at Stage  $n$  there are neighboring fractions  $\frac{a_n}{b_n}$  and  $\frac{c_n}{d_n}$  with  $\frac{a_n}{b_n} < \frac{p}{q} < \frac{c_n}{d_n}$ . What can you say about  $b_n + d_n$  as  $n$  goes to infinity?)

**Solution.** We argue by contradiction. Suppose  $\frac{p}{q}$  is a rational with  $0 < \frac{p}{q} < 1$  that does not appear at any stage of the procedure.

Since  $\frac{0}{1} < \frac{p}{q} < \frac{1}{1}$ , the Stage 1 neighbors enclosing  $\frac{p}{q}$  are  $\frac{a_1}{b_1} = \frac{0}{1}$  and  $\frac{c_1}{d_1} = \frac{1}{1}$ .

Now suppose at Stage  $n$  the fraction  $\frac{p}{q}$  lies strictly between consecutive neighbors  $\frac{a_n}{b_n}$  and  $\frac{c_n}{d_n}$ . At Stage  $n+1$  the Farey sum  $\frac{a_n+c_n}{b_n+d_n}$  is inserted between them. Comparing  $\frac{p}{q}$  to this new fraction, exactly one of the following holds:

- (i)  $\frac{p}{q} = \frac{a_n+c_n}{b_n+d_n}$ . Then  $\frac{p}{q}$  appears at Stage  $n+1$ , contradicting our assumption.
- (ii)  $\frac{p}{q} < \frac{a_n+c_n}{b_n+d_n}$ . The new neighbors of  $\frac{p}{q}$  are  $\frac{a_{n+1}}{b_{n+1}} = \frac{a_n}{b_n}$  and  $\frac{c_{n+1}}{d_{n+1}} = \frac{a_n+c_n}{b_n+d_n}$ , so  $b_{n+1} + d_{n+1} = 2b_n + d_n$ .
- (iii)  $\frac{p}{q} > \frac{a_n+c_n}{b_n+d_n}$ . By symmetry,  $b_{n+1} + d_{n+1} = b_n + 2d_n$ .

Case (i) is ruled out by assumption. In cases (ii) and (iii), since  $b_n \geq 1$  and  $d_n \geq 1$ ,

$$b_{n+1} + d_{n+1} \geq (b_n + d_n) + 1.$$

By induction,  $b_n + d_n \geq (b_1 + d_1) + (n-1) = n+1$  for every  $n \geq 1$ . But Exercise 3 gives  $q \geq b_n + d_n$  for every  $n$ , so  $q \geq n+1$  for all  $n \in \mathbb{N}$ . This is impossible since  $q$  is fixed. The assumption must fail, and  $\frac{p}{q}$  appears at some stage.