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Farey Sums

Number Theory Exercises

The *Farey sum* of fractions $\frac{a}{b}$ and $\frac{c}{d}$ is defined by the formula

$$\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}.$$

The symbol $\oplus$ is a reminder that the Farey sum is not ordinary addition: $\frac{0}{1} \oplus \frac{1}{1} = \frac{1}{2}$, not 1. It also depends on the representatives chosen: $\frac{0}{2} \oplus \frac{1}{1} = \frac{1}{3}$.

Start with $\frac{0}{1}$ and $\frac{1}{1}$. At each stage, insert between every pair of neighboring fractions their Farey sum. The first five stages, with each stage's new fractions in bold:

Stage 1: $\frac{0}{1}, \frac{1}{1}$

Stage 2: $\frac{0}{1}, \mathbf{\frac{1}{2}}, \frac{1}{1}$

Stage 3: $\frac{0}{1}, \mathbf{\frac{1}{3}}, \frac{1}{2}, \mathbf{\frac{2}{3}}, \frac{1}{1}$

Stage 4: $\frac{0}{1}, \mathbf{\frac{1}{4}}, \frac{1}{3}, \mathbf{\frac{2}{5}}, \frac{1}{2}, \mathbf{\frac{3}{5}}, \frac{2}{3}, \mathbf{\frac{3}{4}}, \frac{1}{1}$

Stage 5: $\frac{0}{1}, \mathbf{\frac{1}{5}}, \frac{1}{4}, \mathbf{\frac{2}{7}}, \frac{1}{3}, \mathbf{\frac{3}{8}}, \frac{2}{5}, \mathbf{\frac{3}{7}}, \frac{1}{2}, \mathbf{\frac{4}{7}}, \frac{3}{5}, \mathbf{\frac{5}{8}}, \frac{2}{3}, \mathbf{\frac{5}{7}}, \frac{3}{4}, \mathbf{\frac{4}{5}}, \frac{1}{1}$

The exercises build toward Exercise 4: every rational number between 0 and 1 eventually appears.

Exercise 1. Show that at each stage in the procedure, the fractions appear in increasing order. (Hint: Induction. Compare each inserted fraction with the neighbor on each side.)

Exercise 2. Assume $\frac{a}{b}$ and $\frac{c}{d}$ are neighbors at some stage of the procedure with $\frac{a}{b} < \frac{c}{d}$.

(a) Show $bc - ad = 1$. (Hint: Induction. A pair of neighbors at the next stage has one of two forms.)

(b) Show

$$\frac{c}{d} - \frac{a}{b} = \frac{1}{bd}.$$

Exercise 3. Assume $\frac{a}{b}$ and $\frac{c}{d}$ are as in Exercise 2. Let $\frac{p}{q}$ be a fraction with $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$. Show that $q \geq b + d$.

Hint:

$$\frac{c}{d} - \frac{a}{b} = \left(\frac{c}{d} - \frac{p}{q}\right) + \left(\frac{p}{q} - \frac{a}{b}\right).$$

Exercise 4. Prove that every rational number between 0 and 1 appears somewhere in the procedure. (Hint: Assume for contradiction that $\frac{p}{q}$ does not appear at any stage. So at Stage n there are neighboring fractions $\frac{a_n}{b_n}$ and $\frac{c_n}{d_n}$ with $\frac{a_n}{b_n} < \frac{p}{q} < \frac{c_n}{d_n}$. What can you say about $b_n + d_n$ as n goes to infinity?)