Herndon Math Services
www.herndonmathservices.com

Attracting and Repelling Fixed Points – Solutions

Determining convergence of a recursively defined sequence

Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and let x_0 be a real number. We study the recursively defined sequence

$$x_{n+1} = f(x_n).$$

The sequence (x_n) is called the *orbit* of the initial value x_0. Note that if $x_n \to L$ then $L = f(L)$, so L must be a *fixed point* of f. (To see this, take limits on both sides of $x_{n+1} = f(x_n)$ and apply continuity of f.) Thus, we can think of the fixed point set of f as a collection of "candidate limits" for (x_n). Whether or not the orbit actually converges to any of these candidate limits is determined by the initial value x_0 and by the size of $f'(p)$ at the fixed points p of f.

The *stability criterion* compares $|f'(p)|$ with 1.

Theorem 1 (Stability of fixed points)**.** *Let p be a fixed point of f.*

(i) *If $|f'(p)| < 1$, then p is* attracting*: there is a $\delta > 0$ so that every x_0 with $|x_0 - p| < \delta$ produces an orbit converging to p.*

(ii) *If $|f'(p)| > 1$, then p is* repelling*: there is a $\delta > 0$ so that every x_0 with $0 < |x_0 - p| < \delta$ produces an orbit that eventually leaves $(p - \delta, p + \delta)$.*

If $|f'(p)| = 1$, the criterion is inconclusive.

We explore this in the exercises.

Exercise 1. Let $f(x) = \dfrac{3x - x^3}{2}$ and consider the recursion $x_{n+1} = f(x_n)$.

(a) Find all fixed points of f.

(b) Compute f' and use the stability criterion to classify each fixed point as attracting or repelling.

(c) Show that $x_0 = \frac{1}{2}$ gives $x_n \to 1$ and $x_0 = -\frac{1}{2}$ gives $x_n \to -1$, proving each directly.

(d) Determine the end behavior of (x_n) for each of:

(i) $x_0 = \sqrt{5}$;

(ii) $x_0 = 3$.

(e) The criterion ensures convergence for x_0 sufficiently close to the attracting fixed point 1. Find the largest open interval containing 1 on which every initial value x_0 gives $x_n \to 1$.

Solution. Throughout, $f(x) = \frac{1}{2}(3x - x^3)$, which is odd.

(a) The fixed points solve $f(x) = x \iff x - x^3 = 0 \iff x(1 - x^2) = 0$, so $x = 0, \pm 1$.

(b) Since $f'(x) = \frac{3}{2}(1 - x^2)$, we have $|f'(0)| = \frac{3}{2} > 1$, so 0 is repelling, and $|f'(\pm 1)| = 0 < 1$, so ± 1 are attracting.

(c) Take $x_0 = \frac{1}{2}$. For $x \in (0, 1)$ we have $f(x) = \frac{1}{2}x(3 - x^2) > 0$ and $f(x) - 1 = -\frac{1}{2}(x - 1)^2(x + 2) < 0$, so f maps $(0, 1)$ into itself; also $f(x) - x = \frac{1}{2}x(1 - x^2) > 0$ there, so (x_n) is increasing. Increasing and bounded above by 1, it converges (MCT) to a fixed point in $(0, 1]$, which can only be 1; so $x_n \to 1$. By oddness the orbit from $x_0 = -\frac{1}{2}$ is the negative of this one, so $x_0 = -\frac{1}{2}$ gives $x_n \to -1$. (The repelling point 0 is the basin boundary between the basins of ± 1.)

(d)(i) With $x_0 = \sqrt{5}$: $f(\sqrt{5}) = \frac{1}{2}(3\sqrt{5} - 5\sqrt{5}) = -\sqrt{5}$, and by oddness $f(-\sqrt{5}) = \sqrt{5}$, so the orbit is $\sqrt{5}, -\sqrt{5}, \sqrt{5}, -\sqrt{5}, \ldots$: a 2-cycle, which never converges.

(d)(ii) With $x_0 = 3$: $f(3) = \frac{1}{2}(9 - 27) = -9$, so $|x_1| = 9$. For any x with $|x| \geq 3$ we have $x^2 - 3 \geq 6$, hence

$$|f(x)| = \tfrac{1}{2}\,|x|\,(x^2 - 3) \geq 3\,|x|\,.$$

So once the orbit reaches absolute value at least 3 (already true at x_1), each step at least triples the magnitude: $|x_n| \geq 3^{n-1}\,|x_1|$ for $n \geq 1$. Hence $|x_n| \to \infty$ and the orbit diverges ($3 \mapsto -9 \mapsto 351 \mapsto \cdots$).

(e) The largest open interval containing 1 is $(0, \sqrt{3})$.

On $(0, \sqrt{3})$ we have $f(x) = \frac{1}{2}x(3 - x^2) > 0$, and $f(x) - 1 = -\frac{1}{2}(x - 1)^2(x + 2) < 0$ gives $f(x) < 1$ for $x \neq 1$. So for $x_0 \in (0, \sqrt{3})$ with $x_0 \neq 1$ the first step lands in $(0, 1)$, and by (c) the orbit increases to 1; and $x_0 = 1$ is already the limit. Hence every $x_0 \in (0, \sqrt{3})$ gives $x_n \to 1$.

Both endpoints fail to converge to 1: at the left, $x_0 = 0$ is the repelling fixed point, whose orbit stays at 0; at the right, $f(\sqrt{3}) = 0$, so the orbit of $\sqrt{3}$ lands on 0 and stays there. Any open interval larger than $(0, \sqrt{3})$ must contain 0 or $\sqrt{3}$, so since $1 \in (0, \sqrt{3})$, this is the largest open interval containing 1 on which $x_n \to 1$.

Exercise 2. When $|f'(p)| = 1$ the stability criterion is inconclusive, as demonstrated in the following examples.

(a) Check that 0 is a fixed point of each of

$$f(x) = x - x^3, \qquad g(x) = x + x^3, \qquad h(x) = x + x^2,$$

with $|f'(0)| = |g'(0)| = |h'(0)| = 1$.

(b) Show that 0 is attracting for f, repelling for g, and neither for h.

Conclude that no criterion depending on $|f'(p)|$ alone can decide the borderline case.

Solution. **(a)** Each fixes 0, since $f(0) = g(0) = h(0) = 0$. The derivatives are $f'(x) = 1-3x^2$, $g'(x) = 1+3x^2$, $h'(x) = 1+2x$, so $f'(0) = g'(0) = h'(0) = 1$ and each absolute value is 1.

(b) For $f(x) = x - x^3$: take any x_0 with $0 < |x_0| < 1$ and track the magnitudes $a_n = |x_n|$. Since $|x_n| < 1$ gives $1 - x_n^2 = 1 - a_n^2 > 0$,

$$a_{n+1} = |x_n| \left|1 - x_n^2\right| = a_n(1 - a_n^2) = a_n - a_n^3.$$

This keeps $0 < a_{n+1} < a_n < 1$, so (a_n) decreases and is bounded below by 0; hence $a_n \to L$ for some $L \geq 0$. Letting $n \to \infty$ in $a_{n+1} = a_n - a_n^3$ gives $L = L - L^3$, so $L^3 = 0$ and $L = 0$. Thus $|x_n| \to 0$, i.e. $x_n \to 0$: the point 0 is attracting.

For $g(x) = x + x^3$: again track $a_n = |x_n|$, which now satisfies

$$a_{n+1} = |x_n|\,(1 + x_n^2) = a_n + a_n^3,$$

so (a_n) is increasing. If the orbit stayed inside $(-\delta, \delta)$ for some $\delta > 0$, then (a_n) would be bounded above by δ, hence convergent to some L; letting $n \to \infty$ in $a_{n+1} = a_n + a_n^3$ gives $L = L + L^3$, so $L = 0$. This is impossible, since $a_n \geq a_0 > 0$. So no neighborhood of 0 traps the orbit: 0 is repelling.

For $h(x) = x + x^2 = x(1 + x)$: on $(-1, 0)$ we have $1 + x \in (0, 1)$, so $h(x) < 0$ and $|h(x)| = |x|\,(1+x) < |x|$; the orbit stays in $(-1, 0)$ with $|x_n| \to 0$, so orbits starting just left of 0 converge to it. But for $x > 0$, $h(x) = x + x^2 > x$, so orbits starting just right of 0 move away. Thus not every nearby start converges (so 0 is not attracting) and not every nearby start leaves (so 0 is not repelling): 0 is neither attracting nor repelling.

Since $|f'(0)| = |g'(0)| = |h'(0)| = 1$ yet the three behaviors differ, no criterion using $|f'(p)|$ alone can classify a fixed point with $|f'(p)| = 1$.

Exercise 3. Prove the stability theorem stated in the introduction. You may use the following outline.

(a) (Attracting.) Suppose $|f'(p)| < 1$ and fix a number k with $|f'(p)| < k < 1$. Using continuity of f', show there is a $\delta > 0$ with $|f'(x)| \leq k$ for $|x - p| \leq \delta$, and deduce (by the Mean Value Theorem) that $|f(x) - p| \leq k\,|x - p|$ for such x. Conclude that $|x_0 - p| < \delta$ forces $|x_n - p| \leq k^n\,|x_0 - p|$, and hence $x_n \to p$.

(b) (Repelling.) Suppose $|f'(p)| > 1$. By the same method, now with $|f'(x)| \geq k > 1$ near p, produce a $\delta > 0$ such that every initial value x_0 with $0 < |x_0 - p| < \delta$ must eventually leave $(p - \delta,\, p + \delta)$.

Solution. Recall $f(p) = p$ and f is continuously differentiable.

(a) Fix k with $|f'(p)| < k < 1$. Since f' is continuous and $|f'(p)| < k$, there is a $\delta > 0$ with $|f'(x)| \le k$ for all $x \in [p-\delta, p+\delta]$. For such x the Mean Value Theorem gives a ξ between x and p with $f(x) - p = f(x) - f(p) = f'(\xi)(x-p)$, so

$$|f(x) - p| = |f'(\xi)|\,|x-p| \le k\,|x-p| \qquad \text{whenever } |x-p| \le \delta.$$

Suppose $|x_0 - p| < \delta$. We show $|x_n - p| \le k^n\,|x_0 - p|$ for all $n \ge 0$ by induction. For $n = 0$ this is the equality $|x_0 - p| = k^0\,|x_0 - p|$. Assuming it for n, we have $|x_n - p| \le k^n\,|x_0 - p| \le |x_0 - p| < \delta$, so $x_n \in [p-\delta, p+\delta]$ and the one-step bound applies:

$$|x_{n+1} - p| = |f(x_n) - p| \le k\,|x_n - p| \le k \cdot k^n\,|x_0 - p| = k^{n+1}\,|x_0 - p|\,,$$

which is the claim for $n+1$. Since $0 < k < 1$, $k^n\,|x_0 - p| \to 0$, hence $x_n \to p$.

(b) Fix k with $1 < k < |f'(p)|$. Since f' is continuous and $|f'(p)| > k$, there is a $\delta > 0$ with $|f'(x)| \ge k$ for all $x \in [p-\delta, p+\delta]$; for such x the Mean Value Theorem gives $|f(x) - p| = |f'(\xi)|\,|x-p| \ge k\,|x-p|$. Take any initial value x_0 with $0 < |x_0 - p| < \delta$ and suppose for contradiction that the orbit never leaves $(p-\delta, p+\delta)$, so $|x_n - p| < \delta$ for all n. Then the bound applies at every step, giving $|x_n - p| \ge k^n\,|x_0 - p|$; since $k > 1$ and $|x_0 - p| > 0$, the right-hand side exceeds δ for large n, thus a contradiction. So the orbit must eventually leave $(p-\delta, p+\delta)$.

(The borderline $|f'(p)| = 1$ is left open by both arguments; Exercise 2 shows that it has to be.)