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Attracting and Repelling Fixed Points

Determining convergence of a recursively defined sequence

Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and let x_0 be a real number. We study the recursively defined sequence

$$x_{n+1} = f(x_n).$$

The sequence (x_n) is called the *orbit* of the initial value x_0. Note that if $x_n \to L$ then $L = f(L)$, so L must be a *fixed point* of f. (To see this, take limits on both sides of $x_{n+1} = f(x_n)$ and apply continuity of f.) Thus, we can think of the fixed point set of f as a collection of "candidate limits" for (x_n). Whether or not the orbit actually converges to any of these candidate limits is determined by the initial value x_0 and by the size of $f'(p)$ at the fixed points p of f.

The *stability criterion* compares $|f'(p)|$ with 1.

Theorem 1 (Stability of fixed points)**.** *Let p be a fixed point of f.*

(i) *If $|f'(p)| < 1$, then p is* attracting*: there is a $\delta > 0$ so that every x_0 with $|x_0 - p| < \delta$ produces an orbit converging to p.*

(ii) *If $|f'(p)| > 1$, then p is* repelling*: there is a $\delta > 0$ so that every x_0 with $0 < |x_0 - p| < \delta$ produces an orbit that eventually leaves $(p - \delta, p + \delta)$.*

If $|f'(p)| = 1$, the criterion is inconclusive.

We explore this in the exercises.

Exercise 1. Let $f(x) = \dfrac{3x - x^3}{2}$ and consider the recursion $x_{n+1} = f(x_n)$.

(a) Find all fixed points of f.

(b) Compute f' and use the stability criterion to classify each fixed point as attracting or repelling.

(c) Show that $x_0 = \frac{1}{2}$ gives $x_n \to 1$ and $x_0 = -\frac{1}{2}$ gives $x_n \to -1$, proving each directly.

(d) Determine the end behavior of (x_n) for each of:

   (i) $x_0 = \sqrt{5}$;

(ii) $x_0 = 3$.

(e) The criterion ensures convergence for x_0 sufficiently close to the attracting fixed point 1. Find the largest open interval containing 1 on which every initial value x_0 gives $x_n \to 1$.

Exercise 2. When $|f'(p)| = 1$ the stability criterion is inconclusive, as demonstrated in the following examples.

(a) Check that 0 is a fixed point of each of

$$f(x) = x - x^3, \qquad g(x) = x + x^3, \qquad h(x) = x + x^2,$$

with $|f'(0)| = |g'(0)| = |h'(0)| = 1$.

(b) Show that 0 is attracting for f, repelling for g, and neither for h.

Conclude that no criterion depending on $|f'(p)|$ alone can decide the borderline case.

Exercise 3. Prove the stability theorem stated in the introduction. You may use the following outline.

(a) (Attracting.) Suppose $|f'(p)| < 1$ and fix a number k with $|f'(p)| < k < 1$. Using continuity of f', show there is a $\delta > 0$ with $|f'(x)| \leq k$ for $|x - p| \leq \delta$, and deduce (by the Mean Value Theorem) that $|f(x) - p| \leq k\,|x - p|$ for such x. Conclude that $|x_0 - p| < \delta$ forces $|x_n - p| \leq k^n\,|x_0 - p|$, and hence $x_n \to p$.

(b) (Repelling.) Suppose $|f'(p)| > 1$. By the same method, now with $|f'(x)| \geq k > 1$ near p, produce a $\delta > 0$ such that every initial value x_0 with $0 < |x_0 - p| < \delta$ must eventually leave $(p - \delta,\, p + \delta)$.